1. *Suppose that* $\sum_{n\in\mathbb{Z}} |nC_n| < \infty$, *where* C_n, $n \in \mathbb{Z}$ *are complex numbers. Show that if*

$$f(\theta) := \sum_{n\in Z} C_n e^{in\theta},$$

then

$$f'(\theta) = \sum_{n\in Z} in\hat{f}(n)e^{in\theta}$$

where $\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$.

Solution: The given function f is 2π periodic and its Fourier series is given by $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}$ where

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} C_n e^{in\theta}e^{-in\theta}d\theta = C_n.$$

So we can write $f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta}$. Now let $S_n(\theta) :=) = \sum_{k=-n}^{n} \hat{f}(k)e^{in\theta}$. The derivative S_n' is given as $S_n'(\theta) = \sum_{k=-n}^{n} ik\hat{f}(k)e^{ik\theta}$. Since $\sum_{n\in\mathbb{Z}} |n\hat{f}(n)| < \infty$, $S_n'(\theta)$ converges uniformly. So we have $S_n(\theta)$ converged uniformly to $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta} = f(\theta)$ and $S_n'(\theta)$ converges uniformly to $\sum_{n\in\mathbb{Z}} in\hat{f}(n)e^{in\theta} = g(\theta)$. It implies that $f'(\theta) = g(\theta)$.

$\square$

2. *Let* $\{P_r(\theta), 0 \leq r < 1\}$ *be the Poisson kernel, defined as*

$$P_r(\theta) := \frac{1-r^2}{1+r^2-2rcos\theta}.$$

Show that for each $0 \leq r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1 \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \{P_r(\theta)\}^2 d\theta = \frac{1+r^2}{1-r^2}.$$

Solution: We can directly calculate that

$$\begin{aligned} \frac{1-r^2}{1+r^2-2rcos\theta} &= \frac{(1-re^{i\theta})re^{-i\theta}+1-re^{-i\theta}}{|1-re^{i\theta}|^2} = \frac{re^{-i\theta}}{1-re^{-i\theta}} + \frac{1}{1-re^{i\theta}} \\ &= \sum_{n=1}^{\infty} r^n e^{-ins} + \sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=-\infty}^{-1} r^{-n}e^{in\theta} + \sum_{n=0}^{\infty} r^n e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}. \end{aligned}$$

Note that for $0 \leq r < 1$ the series $\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}$ converges uniformly. So we can interchange the summation and integration. Also it is easy to see that

$$\int_{-\pi}^{\pi} e^{int}dt = \begin{cases} 2\pi, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Now we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = \sum_{n\in\mathbb{Z}} \frac{r^{|n|}}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}d\theta = 1.$$

Using the expression $P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}$ and the Plancherel formula, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \{P_r(\theta)\}^2 d\theta = \sum_{n\in\mathbb{Z}} r^{2|n|} = 1 + 2r^2\frac{1}{1-r^2} = \frac{1+r^2}{1-r^2}.$$

□

3. *For $t > 0$, $x \in \mathbb{R}^d$, let $H(t,x) := \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2}e^{2\pi\xi.x}d\xi$. Then show that*

$$\partial_t H(t,x) = \sum_{j=1}^{d} \partial_j^2 H(t,x), \quad t > 0, x \in \mathbb{R}^d,$$

where ∂_t, ∂_j are the partial derivatives with respect to t and x_j respectively.

Solution: It is well known that the inverse Fourier transform of the Gaussian $e^{-\alpha|x|^2}$ is equal to $\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{|\xi|^2}{4\alpha}}$. Using this we get

$$\int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2}e^{2\pi i\xi.x}d\xi = \frac{1}{2\sqrt{\pi t}}e^{\frac{-|x|^2}{4t}}.$$

Now the only remaining job to prove that $H(t,x) = \frac{1}{2\sqrt{\pi t}}e^{\frac{-|x|^2}{4t}}$ will solve the differential equation $\partial_t H(t,x) = \sum_{j=1}^{d} \partial_j^2 H(t,x)$. we can easily see that

$$\partial_t\left(t^{-1/2}e^{-\frac{|x|^2}{4t}}\right) = \frac{1}{2t\sqrt{t}}e^{-\frac{|x|^2}{4t}} - \frac{|x|^2}{4t^2\sqrt{t}}e^{-\frac{|X|^2}{4t}}$$

and

$$\partial_j\left(t^{-1/2}e^{-\frac{|x|^2}{4t}}\right) = -\frac{x_j}{2t\sqrt{t}}e^{-\frac{|X|^2}{4t}}.$$

Differentiating again with respect to x_j we have

$$\partial_j^2\left(t^{-1/2}e^{-\frac{|x|^2}{4t}}\right) = -\frac{x_j^2}{4t^2\sqrt{t}}e^{-\frac{|X|^2}{4t}} + \frac{1}{2t\sqrt{t}}e^{-\frac{|x|^2}{4t}}.$$

Now a direct substitution will give the required answer. □

4. *Show that the kernel* $k_t(x) := \frac{t}{\pi(t^2+x^2)}, t > 0, x \in \mathbb{R}$ *is an approximate identity.*

Solution: Let $k(x) = (\pi(1+x^2))^{-1}$. We can see that

$$k_t(x) = \frac{t}{\pi(t^2+x^2)} = \frac{1}{t}k(\frac{x}{t}).$$

Since the L^1 norm of k and k_t are same and

$$\int_{\mathbb{R}} \frac{1}{1+x^2}dx = \lim_{x\to\infty}(tan^{-1}(x) - tan^{-1}(-x)) = (\pi/2) - (-\pi/2) = \pi$$

we get $\int_{\mathbb{R}} k_t(x)dx = 1$. Finally for all $\delta > 0$,

$$\frac{1}{\pi}\int_{|x|\geq\delta} \frac{1}{t}\frac{1}{(x/t)^2+1}dx = 1 - \frac{2}{\pi}tan^{-1}(\delta/t) \to 0 \ as \ t \to 0.$$

□

5. *Let* $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the extension to $L^2(\mathbb{R}^d)$ of the Fourier transform on $L^1(\mathbb{R}^d)$ given by the Plancheral theorem. For $f \in L^2(\mathbb{R}^d)$, $a, b \in \mathbb{R}^d$. Compute $\mathcal{F}(f_a)$ and $\mathcal{F}(f^b)$ where $f_a(x) := f(x-a)$ and $f^b(x) := e^{-2\pi b.x}f(x)$.

Solution: The Fourier transform $\mathcal{F}$ of $f \in L^2(\mathbb{R}^d)$ is given by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi ix.\xi}dx.$$

Now by this definition

$$\mathcal{F}(f_a)(\xi) = \int_{\mathbb{R}^d} f(x-a)e^{-2\pi ix.\xi}dx.$$

Apply change of variable $y = x - a$ to the above integral we get that $\mathcal{F}(f_a)(\xi) = e^{2\pi a.\xi}\mathcal{F}(f)(\xi)$. Similarly the Fourier transform of f^b is given by

$$\mathcal{F}(f^b)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi b.x}f(x)e^{-2\pi ix.\xi}dx = \int_{\mathbb{R}^d} f(x)e^{-2\pi ix.(b+\xi)}dx = \mathcal{F}(f)(b+\xi).$$